Density matrix interpretation of solutions of Lie-Nambu equations

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The spectrum of a density matrix $\rho(t)$ is conserved by a Lie-Nambu dynamics if $\rho(t)$ is a self-adjoint and Hilbert-Schmidt solution of a nonlinear triple-bracket equation. This generalizes to arbitrary separable (positive- and indefinite-metric) Hilbert spaces the previous result which was valid for finite-dimensional Hilbert spaces.

I. STATE VECTORS VS. DENSITY MATRICES IN NONLINEAR QUANTUM MECHANICS

There exist prejudices concerning nonlinear generalizations of quantum mechanics. One of them is a belief that any generalization must lead to unphysical effects such as a faster-than-light transfer of information. Although the proofs of these unphysical phenomena are explicit and *mathematically* correct [1–4] they are based on some *physical* assumptions which are unjustified. One of these physically wrong elements is a naive use of the projection postulate. Today we understand that if the dynamics is nonlinear we are not allowed to simply project a solution on some direction in a Hilbert space. One of the reasons is that a projection of a solution is in general not a solution of a nonlinear Schrödinger equation. A more subtle argument is provided by the notion of nonlinear gauge transformations introduced by Doebner and Goldin [5,6] and developed by the Clausthal school [7]. It is clear that there exists a class of nonlinear Schrödinger equations which are obtained by a nonlinear gauge transformation from an ordinary linear Schrödinger equation. They not only give the same probability density in position space but also may look "truely" nonlinear (there exist nonlinear gauge transformations that simply add a nonlinear term but do not alter the kinetic and potential parts in a Hamiltonian). Obviously if the nonexistence theorems were true such "nonlinear" equations would have to lead to unphysical effects. But the point is that they do not lead to *any* new effects since, by definition, they are physically fully equivalent to to the linear theory. Therefore the nonexistence theorems must contain some elements which are physically wrong. From the perspective of the nonlinear gauge transformations it is clear that one of them is a wrong use of the projection postulate. Let us note, however, that the explicit example discussed in [3] is not based on this postulate (and thus is not equivalent to the examples given in [1,2]; a simple argument shows also that the "telegraph" discussed in [3] works in the opposite direction than those from [1,2]). An element which is physically wrong here is the wrong way of describing composite systems. This was clarified by Polchinski [8] and Jordan [9]. The latter work was based on a density matrix reformulation of Weinberg's nonlinear quantum mechanics [10,11].

Density matrices play in nonlinear quantum mechanics a role which is somewhat ambiguous. On the one hand, one of the earliest attempts of formulating a general nonlinear framework for quantum mechanics was Mielnik's "convex formalism" [12]. Its main idea was to keep a figure of states convex and derive a probability interpretation in terms of its global geometric properties. From this perspective the density matrices might be even more fundamental than state vectors. On the other hand, however, all works that start from pure states and nonlinear Schrödinger equations lead to difficulties when it comes to "mixtures". The difficulties are so deep that some authors tend to reject the very notion of a density matrix in a nonlinear context [13], although different proposals of combining mixtures with nonlinearity of pure states exist in the literature (cf. [14,15]).

A nonlinear extension of quantum mechanics based on a triple bracket Nambu-type generalization of the Liouville-von Neumann equation (cf. [16]) proposed by one of us [17–19] starts from a completely different perspective. The idea is to find a general scheme which on one hand includes the linear and Weinberg-Bona-Jordan cases and on the other leaves some room for nonlinear generalizations that do not use nonlinear Hamiltonians. Such a starting point is motivated by ambiguities in probability interpretation caused by the notion of eigenvalue of a nonlinear operator [20]. The Lie-Nambu scheme proved very powerful and elegant and has, in our oppinion, several advantages over the standard paradigm of nonlinear Schrödinger equations. The density matrices play in this formalism a fundamental role. Still the basic question of an interpretation of solutions $\rho(t)$ as density matrices was not fully clarified in the earlier work. The Theorem 5 discussed in [18] worked essentially in finite dimensional cases whereas the generic infinite dimensional Hilbert space problem was left open. In this paper we give an alternative proof which generalizes this theorem to infinite dimensional Hilbert spaces. We consider also the indefinite-metric case which is of some interest for a relativistic theory.

II. TRIPLE-BRACKET EQUATIONS AND SPECTRAL PROPERTIES OF THEIR SOLUTIONS

Consider a one-parameter family $\rho = \rho(t)$, $t \in \boldsymbol{R}$, of Hilbert-Schmidt self-adjoint operators acting in a separable Hilbert space and satisfying the Lie-Nambu equation

$$i\dot{\rho}_a = \{\rho_a, H, S\}. \tag{1}$$

Here $\rho_a := \rho_{AA'}(\boldsymbol{a}, \boldsymbol{a}')$ are components of ρ in some basis, A and A' are discrete (say, spinor) indices and $\boldsymbol{a}$, $\boldsymbol{a}'$ the continuous ones (coresponding to, say, position or momentum). The dot represents a derivative with respect to the parameter t. In nonrelativistic case this is just an ordinary time. In the relativistic case the meaning of t depends on a formalism (t is time in some reference frame in [18], and a "proper time" in the off-shell formulation given in [19]). $H = H(\rho)$ is any (functionally) differentiable functional of ρ and $S = S(\rho) = S\big(C_1(\rho), \dots, C_k(\rho), \dots\big)$ is differentiable in $C_n(\rho)$ (see Appendix V B). We assume the following summation convention for the composite indices a: A contraction of two composite indices means simultaneous summation over the discrete indices and integration (with respect to an appropriate measure) of the continuous ones. The triple bracket itself is defined as

$$\{F, G, H\} = \Omega_{abc} \frac{\delta F}{\delta \rho_a} \frac{\delta G}{\delta \rho_b} \frac{\delta H}{\delta \rho_c} \tag{2}$$

where Ω_{abc} are structure constants of an infinite-dimensional Lie algebra which also depends on the model [18,19] (see Appendix V A). The indices in Ω_{abc} can be raised and lowered by a metric discussed in detail in [18,19]. The metric is well defined for both finite- and infinite-dimensional Lie algebras and is not equivalent to the Cartan-Killing one (the latter does not exist in the infinite-dimensional case).

Before we proceed with the main theorem we shall first prove a few useful technical results.

Lemma 1: Let $\{p_k\}_{k=1}^{\infty}$ be a sequence of nonnegative numbers such that $p_1 \geq p_2 \geq p_3 \geq \dots$ and the series $\sum_{k=1}^{\infty} p_k$ is convergent. Then

$$\lim_{m\to\infty} \left(\sum_{k=1}^{\infty} p_k^m \right)^{\frac{1}{m}} = p_1.$$

Proof. Using the three sequences theorem one immediately generalizes the standard proof known for finite sequences. □

Lemma 2: Let $\{p_k\}_{k=1}^{\infty}$, $\{q_k\}_{k=1}^{\infty}$ be two sequences fulfiling the assumptions of the above lemma. Suppose that for every $m \in \mathbf{N}$

$$\sum_{k=1}^{\infty} p_k^m = \sum_{k=1}^{\infty} q_k^m.$$

Then $p_k = q_k$ for every $k = 1, 2, \dots$.

Proof. The equality $p_1 = q_1$ follows directly from Lemma 1. The proof is completed by induction.□

Lemma 3: Let $\{q_k\}_{k=1}^{\infty}$, $q_k \in \mathbf{R}$, be an arbitrary sequence and $\{p_k\}_{k=1}^{\infty}$ satisfy the assumptions of Lemma 1. Suppose that for every $m \in \mathbf{N}$

$$\sum_{k=1}^{\infty} p_k^m = \sum_{k=1}^{\infty} q_k^m.$$

Then $\{p_k\}_{k=1}^{\infty} = \{q_k\}_{k=1}^{\infty}$ up to permutation.

Proof. Define two sequences $\{\tilde{p}_k\}_{k=1}^{\infty}$, $\{\tilde{q}_k\}_{k=1}^{\infty}$, where $\tilde{p}_k = p_k^2$, $\tilde{q}_k = q_k^2$. $\{\tilde{q}_k\}_{k=1}^{\infty}$ is absolutely convergent so we can rearrange its terms in such a way that the rearranged sequence satisfies assumptions of Lemma 1. Assume this done. Lemma 2 implies that $|q_k| = p_k$, for any k. Let Q_+ be the subset of positive elements of $\{q_k\}_{k=1}^{\infty}$. Q_+ is non-empty since otherwise all elements of $\{q_k\}_{k=1}^{\infty}$ would be ≤ 0 which contradicts the assumption that $\sum_{k=1}^{\infty} q_k = \sum_{k=1}^{\infty} p_k > 0$. (We exclude the trivial case where all $p_k = 0$.) Therefore for any $q_k \in Q_+$ there exists p_k such that $p_k = q_k$. Subtracting these elements from both sides of

$$\sum_{k=1}^{\infty} p_k^m = \sum_{k=1}^{\infty} q_k^m$$

we get for $m = 1$

$$\sum_{\text{remaining}\, p_k} p_k = \sum_{q_k \leq 0} q_k$$

which can hold if and only if all such $q_k = 0$. Therefore all $q_k \geq 0$ and $p_k = q_k$ for any k.□

We are interested in solutions of (1) where $\rho(t)$ are Hilbert-Schmidt self-adjoint operators acting in a separable Hilbert space. The following theorem states that spectrum of $\rho(t)$ is conserved by the Lie-Nambu dynamics.

Theorem 4: Let $\rho(t)$ be a Hilbert-Schmidt self-adjoint solution of (1) whose spectrum is $\operatorname{sp}\rho(t) = \{\lambda_k(t)\}_{k=1}^{\infty}$. If $\{\lambda_k(0)\}_{k=1}^{\infty}$ satisfies assumptions of Lemma 1 then $\lambda_k(t) = \lambda_k(0)$ for any t.

Proof: For any t the solution can be written as $\rho_a(t) = \sum_{k=1}^{\infty} \lambda_k(t) \phi_A^k(t, \boldsymbol{a}) \bar{\phi}_{A'}^k(t, \boldsymbol{a}')$. According to Theorem 4 in [18] the functional $C_n(\rho)$, $n \in \mathbf{N}$, (see Appendix V B) is time independent. Therefore

$$\sum_{k=1}^{\infty} (\eta^{kk})^n \lambda_k(t)^n = \sum_{k=1}^{\infty} (\eta^{kk})^n \lambda_k(0)^n \tag{3}$$

for any t and n (η^{kk} includes the indefinite metric case, see Appendix V B). Lemma 3 implies that the sequences $\{\lambda_k(t)\}_{k=1}^{\infty}$ and $\{\lambda_k(0)\}_{k=1}^{\infty}$ are identical up to permutation if the metric η^{kl} is positive definite. In the indefinite metric case we define $\tilde{\lambda}_k(t) = \lambda_k(t)^2$. (3) implies

$$\sum_{k=1}^{\infty} \tilde{\lambda}_k(t)^n = \sum_{k=1}^{\infty} \tilde{\lambda}_k(0)^n.$$

Lemma 3 again implies that the sequences $\{|\lambda_k(t)|\}_{k=1}^{\infty}$ and $\{\lambda_k(0)\}_{k=1}^{\infty}$ are identical up to permutation. Finally continuity in t means that $\lambda_k(t) = \lambda_k(0)$ in both cases.$\Box$

III. CONVEXITY PRINCIPLE: AN EXAMPLE

A nonlinearly evolving density matrix cannot satisfy an ordinary convexity principle. Still, being a Hilbert-Schmidt operator it can be spectrally decomposed and the spectral projectors can be regarded as its "pure state components". This is justified by the fact that the spectrum of $\rho(t)$ is time-independent. To see how this works consider a simple example. The example simultaneously illustrates the peculiarity of the triple-bracket formalism: An existence of nonlinearities that become invisible on pure states. Let $H(\rho) = \mathrm{Tr}\,(h\rho)$ where h is a 2×2 Hermitian matrix, $\rho = \rho_0\mathbf{1} + \boldsymbol{\rho}\cdot\boldsymbol{\sigma}$, and

$$S(\rho) = \frac{2}{3}\big(\mathrm{Tr}\,(\rho)\mathrm{Tr}\,(\rho^3)\big)^{1/2}. \tag{4}$$

(4) is the "entropy" S_3 given by Eq. (75) in [18]. The Lie-Nambu equation (1) is now equivalent to the matrix equation

$$i\dot{\rho} = \Big[\frac{\mathrm{Tr}\,(\rho)}{\mathrm{Tr}\,(\rho^3)}\Big]^{1/2}[h,\rho^2]. \tag{5}$$

Its solution normalized by $\mathrm{Tr}\,\rho = 1$ is

$$\rho(t) = \frac{1}{2}\mathbf{1} + \exp\left[\frac{-iht}{\sqrt{\frac{1}{4}+3\boldsymbol{\rho}^2}}\right]\boldsymbol{\rho}\cdot\boldsymbol{\sigma}\exp\left[\frac{iht}{\sqrt{\frac{1}{4}+3\boldsymbol{\rho}^2}}\right]. \tag{6}$$

For $h = E\sigma_1$ and $\boldsymbol{\rho}\cdot\boldsymbol{\sigma} = \frac{\varepsilon}{2}\sigma_3$ we find $\lambda_1 = (1+\varepsilon)/2$, $\lambda_2 = (1-\varepsilon)/2$, and

$$\phi_A^1 = \begin{pmatrix}\cos\omega(E,\varepsilon)t\\ -i\sin\omega(E,\varepsilon)t\end{pmatrix}, \quad \phi_A^2 = \begin{pmatrix}-i\sin\omega(E,\varepsilon)t\\ \cos\omega(E,\varepsilon)t\end{pmatrix},$$

where $\omega(E,\varepsilon) = 2E/\sqrt{1+3\varepsilon^2}$. Notice that the vectors ϕ_A^k, $k = 1,2$, depend on λ_k. In the linear case we can solve equations for orthogonal pure states and then form their convex combinations with coefficients λ_k, which in no way affects the form of the pure states that form the mixture. In the nonlinear case the "pure state" components of the mixture *do* depend on the coefficients λ_k [19]. The dynamics of $\rho(t)$ is nonlinear even though the Hamiltonian is given by the linear operator h. A possibility of introducing nonlinearities without modifications of an algebra of observables is one of the important differences between the Lie-Nambu formalism and nonlinear Schrödinger equations. For $\boldsymbol{\rho}^2 = 1/4$ the density matrix is a projector and its dynamics is linear.

Now consider a solution of (1) which at $t = 0$ is a convex combination

$$\rho(0) = p_1\rho_1(0) + p_2\rho_2(0) \tag{7}$$

of two not necessarily mutually orthogonal density matrices. Let $\rho_1(0) = \rho_{10}\mathbf{1} + \boldsymbol{\rho}_1\cdot\boldsymbol{\sigma}$ and $\rho_2(0) = \rho_{20}\mathbf{1} + \boldsymbol{\rho}_2\cdot\boldsymbol{\sigma}$. The solution we look for is given by (6) but with $\boldsymbol{\rho} = p_1\boldsymbol{\rho}_1 + p_2\boldsymbol{\rho}_2$. The Hilbert-Schmidt vectors ϕ_A^k depend now not only on the eigenvalues of $\rho_1(0)$, $\rho_2(0)$, but also on p_1 and p_2. This implies also that the dynamics of the density matrix can be written here as

$$\rho(t) = p_1U\rho_1(0)U^\dagger + p_2U\rho_2(0)U^\dagger \tag{8}$$

where $U = U(t,\rho_1(0),\rho_2(0),p_1,p_2)$ is unitary but parametrized by the initial condition.

Typically a nonlinear evolution is a result of a mean-field-type averaging procedure. The example shows that there may exist another mechanism: Nonlinearity via an entanglement [21]. Indeed a subsystem may start to evolve nonlinearly if its reduced density matrix evolves from a pure state to a mixture, and this happens whenever the subsystem gets entangled with another one.

IV. POSITIVITY VS. COMPLETE POSITIVITY

We have shown that self-adjoint Hilbert-Schmidt solutions of nonlinear Lie-Nambu equations are positive for $t \neq 0$ if they are positive at $t = 0$. Typically it is assumed that density matrices should be described by *completely* positive maps. A physical motivation behind complete positivity is the problem of extension of dynamics from subsystems to composite systems [22]: One requires that a dynamics of a system described by ρ should allow to treat this system as a trivially embedded subsystem of a bigger one. The evolution of the bigger system should preserve positivity of *its* density matrix. If this is the case, and if for any t the density matrix of the composite system is positive independently of the dimension of the system we have added, then the dynamics of the original system is said to be completely positive. One additional technical assumption one makes is finite dimensionality of the system one adds.

In the standard analysis of completely positive maps one assumes the maps are linear. But the dynamics we have discussed in this Letter is nonlinear. It can be shown that the definition of a nonlinear completely positive map introduced in mathematical literature [23,24] and applied to Hartree-type equations in [26,25] is physically incorrect. This problem is analyzed elsewhere [27–29]. From the point of view of this Letter it is sufficient to note that there exists a large class of nonliner triple-bracket equations that do not lead to any problem with extension of dynamics from subsystems to composite systems. They satisfy all *physical* requirements typically associated with the notion of a completely positive map. Quite surprisingly, they are not in the form one takes as a departure point for the discussion of complete positivity of nonlinear maps in the mathematical literature.

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V. APPENDICES

A. Structure constants

Consider a Hilbert space of vectors ψ_α where the abstract index α can be discrete, continuous, or composite [18]. Denote the scalar product by $\langle\phi,\psi\rangle = \omega^{\alpha\alpha'}\phi_\alpha\bar{\psi}_{\alpha'}$. The tensor $\omega^{\alpha\alpha'}$ is in general a distribution whose inverse is $I_{\alpha\alpha'}$. By the inverse it is meant that

$$\omega^{\alpha\alpha'}I_{\alpha\beta'} = \delta_{\beta'}{}^{\alpha'} \tag{9}$$

$$\omega^{\alpha\alpha'}I_{\beta\alpha'} = \delta_{\beta}{}^{\alpha} \tag{10}$$

where the δ's mean the Kronecker or Dirac deltas, or their products. In a Hamiltonian formulation of Schrödinger-type equations $\omega^{\alpha\alpha'}$ and $I_{\alpha\alpha'}$ play the roles of a symplectic form and a Poisson tensor, respectively [18,19]. The structure constants are

$$\Omega^a{}_{bc} = \delta_{\beta'}{}^{\alpha'}\delta_{\gamma}{}^{\alpha}I_{\beta\gamma'} - \delta_{\gamma'}{}^{\alpha'}\delta_{\beta}{}^{\alpha}I_{\gamma\beta'} \tag{11}$$

$$\Omega_{abc} = I_{\alpha\beta'}I_{\beta\gamma'}I_{\gamma\alpha'} - I_{\alpha\gamma'}I_{\beta\alpha'}I_{\gamma\beta'} \tag{12}$$

$$\Omega^{abc} = -\omega^{\alpha\beta'}\omega^{\beta\gamma'}\omega^{\gamma\alpha'} + \omega^{\alpha\gamma'}\omega^{\beta\alpha'}\omega^{\gamma\beta'}. \tag{13}$$

Notice the spinor-type convention we use. Different equations (nonrelativistic Schrödinger, positive-metric Dirac, off-shell Dirac, Bargmann-Wigner) correspond to different Hilbert spaces, ω's and I's but the form of the structure constants is always the same. The indices are raised and lowered by metric tensors defined below.

B. Metric tensors and Casimir invariants

We define higher-order metric tensors by

$$g^{a_1\dots a_n} = \omega^{\alpha_1\alpha'_n}\omega^{\alpha_2\alpha'_1}\omega^{\alpha_3\alpha'_2}\dots\omega^{\alpha_{n-1}\alpha'_{n-2}}\omega^{\alpha_n\alpha'_{n-1}} \tag{14}$$

$$G_{a_1\dots a_n} = I_{\alpha_1\alpha'_n}I_{\alpha_2\alpha'_1}I_{\alpha_3\alpha'_2}\dots I_{\alpha_{n-1}\alpha'_{n-2}}I_{\alpha_n\alpha'_{n-1}} \tag{15}$$

$$= g_{a_n\dots a_1}. \tag{16}$$

For $n=1$ we get just the symplectic form and the Poisson tensor. For $n=2$ we obtain the metric tensor on the Lie algebra — it is this tensor that lowers and raises the indices in the structure constants and (14), (15). For higher n's the tensors define higher order Casimir invariants of the Lie-Nambu bracket

$$C_n(\rho) = g^{a_1\dots a_n}\rho_{a_1}\dots\rho_{a_n} = g_{a_1\dots a_n}\rho^{a_1}\dots\rho^{a_n} \tag{17}$$

where $\rho_a = \rho_{\alpha\alpha'}$. Let $\rho_a = \sum_{k=1}^\infty \lambda_k\phi^k_\alpha\bar{\phi}^k_{\alpha'}$ be the Hilbert-Schmidt decomposition of ρ. The vectors ϕ^k_α are orthonormal ($\omega^{\alpha\alpha'}\phi^k_\alpha\bar{\phi}^l_{\alpha'} = \eta^{kl} = \eta^{kk}\delta^{kl}$; $\eta^{kk}=\pm1$ if the metric is not positive definite) which implies that $C_n(\rho) = \sum_{k=1}^\infty(\eta^{kk})^n\lambda_k^n = \mathrm{Tr}\,(\rho^n)$. The metric tensors are therefore an abstract index counterpart of trace.

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